Option pricing with imprecisely stated volatility: A fuzzy-random approach[1]

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Abstract
The option pricing model performance crucially depends on the ability to estimate all necessary input parameters successfully. Within the standard models of Black-Scholes type, the most important parameter is volatility. Since it is often very difficult to obtain a single number, an alternative can be to apply interval approach or more generalized fuzzy-random approach. In this paper recent knowledge of fuzzy numbers and their approximations is utilized in order to suggest fuzzy-random simulation approach to option price modeling, ie. we use fuzzy-random variables. In particular, we suggest to replace a crisp volatility parameter in the standard market model (ie. Black-Scholes type) by a fuzzy random variable, which can be easily evaluated by Monte Carlo simulation. Application possibilities are shown on illustrative examples. In particular, we evaluate the model for various input data and option types. The results are compared to the Black-Scholes option price and market option prices.

Keywords: fuzzy random variable, plain vanilla option, Monte Carlo simulation
JEL codes: C49, C58

1. Introduction

A standard approach to option pricing is based on Black-Scholes type (BS hereafter) models (Black and Scholes, 1973) utilizing the no-arbitrage argument of complete markets. However, there are three crucial assumptions – the option underlying log-returns follow normal distribution, there is unique and deterministic riskless rate as well as the volatility of underlying log-returns – that must be fulfilled, otherwise the BS models might provide false results.

Especially the suggestion of the normality assumption is quite surprising, since the non-normality of asset price returns used to be a well-known fact at least starting with the empirical studies of Mandelbrot (1963a,b) and Fama (1965). It is therefore natural, that there have arisen many alternative models taking into account also the real (or risk neutral) features of market returns, namely

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skewness and kurtosis, by adding more parameters to the original model – a very comprehensive review provide eg. Jondeau et al. (2006) or Cont and Tankov (2010).

Moreover, while riskless rate assumption can be acceptable at well predictable markets with high liquidity (ie. we can use forward rates related to the option maturity), the volatility of returns is a quantity, which is neither directly observable nor tradable. Usually, the only way how to get the instantaneous volatility is to invert option pricing model (a so called *implied volatility*).

It therefore follows that the BS price at one time moment can be related to a whole set of implied volatilities as given by various maturity and moneyness of tradable options and results into implied volatility curve or surface (a so called *smirk* or *smile*), see eg. Rebonato (2004).

One way, how to cope with a volatility parameter is to suppose its stochastic nature, which can lead, under some additional assumptions (especially the normality of log-returns) even to a closed-form formula to option pricing (see eg. Heston (1993) and Heston and Nandi (2000)). Alternatively, instead of specifying a probability distribution of the volatility parameter, we can adopt an interval approach based on extremal observations of volatility in the past or a so called *worst-case volatility scenario approach* (see Buff (2002) for more details). The drawback of the interval approach is that we have only the boundaries.

Our aim in this paper is to formulate a fuzz-random process for the volatility parameter and evaluate selected options via Monte Carlo simulation. We show that it is very intuitive to extend the interval approach by describing the boundaries via fuzzy random numbers. Although several authors have already provided ‘analytical’ formulas (see eg. Yoshida, 2003) and others preferred lattice approximations (see eg. Zmeškal, 2001), due to our knowledge a simulation approach has not been utilized yet. In this context we recall, that Monte Carlo simulation is a very useful approach eg. in case of complex underlying processes and/or payoff functions (see eg. Boyle et al., 1997)

We proceed as follows. In the following section we briefly describe the standard approach to option pricing via Monte Carlo simulation in line with Tichý (2010). Then, we suggest several ways how to deal with the option pricing problem under various assumptions about the volatility. Next, we provide the definition of fuzzy random variables – we follow the approach adopted recently by Holčapek and Tichý (2011) – and also formulate a fuzzy-random process. Finally, we provide an empirical study assuming an option on German stock index (DAX).

2. Standard approach

Let us denote the underlying asset price at maturity time as S_T and the exercise price as K. Then we can denote the payoff function Ψ for European call ($p=1$) and put ($p=-1$) options as $\left(p\left(S_T-K\right)\right)^+$ with $(x)^+ \equiv \max(x,0)$. For the option value at time $t<T$ it generally holds that:

$$f_t = e^{d_\tau}\mathrm{E}[\Psi_T^{vanilla\,call/put}] = e^{d_\tau}\mathrm{E}[\left(p\left(S_T-K\right)\right)^+], \tag{1}$$

where a discount factor d_τ relates to the probability measure under which the expectation operator E is evaluated and $\tau = T-t$ denotes the remaining time to maturity. Commonly, E^{Π} denotes the real world expectation (under physical probability measure), while E^{Θ} is used within the risk-neutral world, ie. $e^{-r\tau}\mathrm{E}^{\Theta}[S_T] = S_t$, where r is a riskless rate valid over time interval τ.

Since financial asset prices are often restricted to positive values only, geometric processes are commonly preferred. If, for example, $Z(t)$ denotes a stochastic process for log-returns of financial asset S, eg. a non-dividend paying stock, in order to model its price in time we have to evaluate the exponential function of $Z(t)$. It follows that under Θ formula (1) can be rewritten into (assuming $p=1$):

$$f_t = e^{-r\tau}\mathrm{E}^{\Theta}\left[\left(S_t e^{r\tau + Z_\tau^{\Theta}} - K\right)^+\right], \tag{2}$$

where Z_τ^{Θ} is a (potentially compensated) realization of a suitable stochastic process over τ such that it is ensured that $\mathrm{E}^{\Theta}[S_t e^{Z_\tau^{\Theta}}]$ is a martingale.

The optimal choice of $Z_\tau^{\mathbb{Q}}$ depends on the assumptions (observations) about the returns of the underlying asset. If the process is sufficiently tractable, (2) can be solved analytically leading to a closed form formula, see eg. risk-neutral derivation of the Black-Scholes model in any textbooks on quantitative finance. Alternatively, we can utilize the law of large numbers and evaluate the expectation in (2) via Monte Carlo simulation, ie. sufficiently large number N of independent scenarios is taken from the relevant probability distribution of Z_τ^{Q} (see eg. Glasserman (2004) for more details):

$$f_t = e^{-r\tau}\mathrm{E}^{\mathbb{Q}}\left[\left(S_t e^{r\tau+Z_\tau^{\mathbb{Q}}} - K\right)^+\right] \approx \frac{e^{-r\tau}}{N}\sum_{i=1}^{N}\left(S_t e^{r\tau+Z_\tau^{\mathbb{Q}(i)}} - K\right)^+, \tag{3}$$

where superscript (i) refers to i-th scenario from a given probability space.

Within the BS model, $Z_\tau^{\mathbb{Q}}$ is the zero-drift diffusion process based on standard normal variates ε:

$$Z_\tau^{\mathbb{Q}} = \sigma\sqrt{\tau}\varepsilon - \frac{\sigma^2}{2}\tau\ . \tag{4}$$

Here, σ is used to scale the variance (and volatility) of underlying returns and the last term ensures that the exponential of (4) is martingale (we will use only ω from now).

Since it is often difficult to select a one number to be put into (4), it can be replaced by a stochastic process – however, suitable candidates should respect the empirical features of variance (volatility) process, ie. positive values and mean-reverting tendency. Obviously, if the information available about the source of uncertainty are not sufficient to select a reliable candidate for its stochastic evolution, one can prefer to replace $Z_\tau^{\mathbb{Q}}$ by a fuzzy-stochastic variable. We will suggest several alternative models in the following section.

3. Option pricing models and assumptions about the volatility

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3.1 Standard market model

Due to its overwhelmed usage, we call the model adopted by Black and Scholes in their seminal paper (Black and Scholes, 1973) for the evolution of the underlying asset price as the standard market model:

$$Z(t) = \sigma\sqrt{t}\varepsilon - \omega t, \tag{5}$$

Here, ε is a random term from standard normal distribution and σ is constant parameter, the volatility of the underlying asset price returns, and ω is a mean correcting parameter, see (4).

3.2 Implied volatility approach

In the standard market model above, at least two assumptions are not fulfilled by real market conditions – the returns do not follow Gaussian distribution and the volatility is not constant over time. Thus, a common practice is to invert the BS option pricing formula and obtain a so called implied volatility for a set of traded maturities and exercise prices. Next, the implied volatility can be used for valuation of OTC derivatives. However, it can happen that OTC derivatives are of different parameters than those traded at the market. In this case, it can be useful to estimate a function relationship among the implied volatility, maturity, moneyness and possibly also payoff function – see eg. Rebonato (2004)

3.3 Stochastic volatility approach

Another direction of research is focused on replacing a constant parameter of volatility σ by a suitable stochastic process. For example, Heston (1993) provided a closed-form formula for option pricing problem assuming that variance follows:

$$\sigma(t) = \sqrt{v(t)} \rightarrow dv = \xi(\bar{v} - v_t)dt + \phi\sqrt{v_t}\varepsilon_2. \tag{6}$$

Here, ε_2 is different, and potentially dependent to ε in (5), $\bar{v}$ describes the long-run average of variance, ϕ is its volatility and ξ allows one to calibrate the velocity of the reversion to $\bar{v}$. Obviously, we can assume any other process, which will provide us positive values of volatility (variance) and will possibly take into account its mean-reverting tendency. In this case, however, it can be inevitable to run a Monte Carlo simulation.

3.4 Interval approach

The stochastic volatility approach requires the ability to formulate a suitable stochastic process that is followed by volatility or variance in time, including reliable estimates to its parameters. Quite often, however, we cannot be sure how relevant is the information about the past for the future. One possibility is to use only the maximal and minimal values we have observed, ie. we use boundaries for the volatility evolution:

$$Z(t) = \sigma\sqrt{t}\varepsilon - \omega t \mid \sigma_{\min} \leq \sigma \leq \sigma_{\max}. \tag{7}$$

which provides us an interval for the option price.

4.5 Volatility as a fuzzy parameter

Notwithstanding, we can make a one step further and define a distinct degree of reliability for particular boundaries. Clearly, the resulting option price will not be an interval as in (7), but rather a whole set of intervals.

Let σ_{LU} be an LU-fuzzy number (see the next section) defined around the crisp estimation of σ. Then we can model price returns by the following fuzzy-stochastic model:

$$Z(t) = \sigma_{LU}\sqrt{t}\varepsilon - \omega_{LU}t. \tag{8}$$

Obviously, σ_{LU} since is LU-fuzzy number, it is inevitable to define also the mean correcting parameter ω as the LU-fuzzy number. Next, we can extend (8) so that σ_{LU} is not defined around deterministic estimation of σ but rather around random number from uniform distribution. We will provide the foundations of this approach in the next section.

4. Fuzzy number and fuzzy random variable

Let $\mathbb{R}$ denotes the set of real numbers and $A : \mathbb{R} \rightarrow [0,1]$ be a mapping. We say that A is a *fuzzy number* if A is normal (ie. there exits an element x_0 such that $A(x_0) = 1$), convex (ie. $A(\lambda x + (1-\lambda)y) \geq \min(A(x), A(y))$ for any $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$), upper semicontinuous and $\mathrm{supp}(A)$ is bounded, where $\mathrm{supp}(A) = cl\{x \in \mathbb{R} \mid A(x) > 0\}$ and *cl* is the closure operator (see Dubois and Prade, 1978). Note that the most popular models of fuzzy numbers are the triangular and trapezoidal shaped models investigated by Dubois and Prade (1980). Their popularity follows from the simple calculus as addition or multiplication of fuzzy numbers which can be established for them. This is also a reason why we can find many recent papers on the approximation of fuzzy numbers by the aforementioned models (see eg. Ban (2009a,b) and the references therein). In order to model fuzzy numbers we will use a more advanced model of fuzzy numbers based on the interpolation of given knots using rational splines that was proposed by Guerra and Stefanini (2005) and developed by Stefanini et al. (2006). This model generalizes triangular fuzzy numbers and provides a broad variety of shapes enabling more precise representation of fuzzy real data. Nevertheless, the calculus remains very simple.

Recall that a piecewise rational cubic Hermite parametric function $P \in C^1[\alpha_0, \alpha_n]$, with parameters v_i, w_i, $i = 0, \ldots, n-1$, is defined for $\alpha \in [\alpha_i, \alpha_{i+1}]$, $i = 0, \ldots, n-1$ by

$$P(\alpha) = P_i(\alpha, v_i, w_i) =$$
$$\frac{(1-\theta)^3 f_i + \theta(1-\theta)^2(v_i f_i + h_i d_i) + \theta^2(1-\theta)(w_i f_{i+1} - h_i d_{i+1}) + \theta^3 f_{i+1}}{(1-\theta)^3 + v_i\theta(1-\theta)^2 + w_i\theta^2(1-\theta) + \theta^3},$$

where the notations f_i and d_i are, respectively, the real data values and the first derivative values (slopes) at the knots $\alpha_0 < \cdots < \alpha_n$, $h_i = \alpha_{i+1} - \alpha_i$, $\theta = (\alpha - \alpha_i)/h_i$ and $v_i, w_i \geq 0$. The parameters v_i and w_i are called the tension parameters. In this work, we will use a global monotonicity setting (cf. Stefanini et al., 2006):

$$v_i = w_i = \begin{cases} \dfrac{d_{i+1} + d_i}{f_{i+1} - f_i}, & if \;\; f_{i+1} \neq f_i; \\ 0, & otherwise. \end{cases} \tag{9}$$

A main reason for this assumption is a natural calculus which can be introduced for fuzzy numbers based on this type of parametric functions. One can see that each such parametric function $P \in C^1[\alpha_0, \alpha_n]$ may be expressed in the matrix form consisting of parameters as follows:

$$\mathbf{P} = \begin{pmatrix} \mathbf{f} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} f_{\alpha_0} & \cdots & f_{\alpha_n} \\ d_{\alpha_0} & \cdots & d_{\alpha_n} \end{pmatrix} \tag{10}$$

for a partition $\alpha_0 < \cdots < \alpha_n$ of the interval $[a_0, a_n]$.

In what follows, we will identify each parametric function $P \in C^1[\alpha_0, \alpha_n]$ satisfying the presumption on v_i and w_i given in (9) is satisfied, with a matrix $\mathbf{P}$ established above and simply write $P(\alpha) = \mathbf{P}(\alpha)$. Now we may define a special case of LU-fuzzy numbers introduced in Guerra and Stefanini (2006). Note that our definition is slightly different than the original one, but the idea remains the same. Recall that an α-cut of a fuzzy number A is a set $A_\alpha = \{x \in \mathbb{R} \mid A(x) \geq \alpha\}$. A fuzzy number A is an *LU-fuzzy number*, if there exist a partition $0 = a_0 < \cdots < \alpha_n = 1$ and $2 \times (n+1)$ matrices $\mathbf{A}^-$ and $\mathbf{A}^+$ (in the form of (10)) such that

1. $A_\alpha = [\mathbf{A}^-(\alpha), \mathbf{A}^+(\alpha)]$ for any $\alpha \in [0,1]$,
2. $f^-_{\alpha_{i+1}} \geq f^-_{\alpha_i}$ and $f^+_{\alpha_{i+1}} \leq f^+_{\alpha_i}$ for any $i = 0, \ldots, n-1$, and $f^-_{\alpha_n} = f^+_{\alpha_n}$,
3. $d^-_{\alpha_i} \geq 0$ and $d^+_{\alpha_i} \leq 0$ for any $i = 0, \ldots, n$,
4. if $f^-_{\alpha_i} = f^-_{\alpha_{i+1}}$ (or $f^+_{\alpha_i} = f^+_{\alpha_{i+1}}$), then $d^-_{\alpha_i} = d^-_{\alpha_{i+1}}$ (or $d^+_{\alpha_i} = d^+_{\alpha_{i+1}}$).

An LU-fuzzy number will be denoted by $\mathbf{A} = (\mathbf{A}^-, \mathbf{A}^+)$ and the set of all LU-fuzzy numbers defined over a partition $0 = a_0 < \cdots < \alpha_n = 1$ will be denoted by $\mathcal{F}_{LU}(\alpha_0, \ldots, \alpha_n)$.

Let $D \subseteq \mathbb{R}^m$ and $g : D \to \mathbb{R}$ be a real function which has all partial derivatives on the domain D, ie. $g'_{x_k}(a_1, \ldots, a_m) \in \mathbb{R}$ for any $(a_1, \ldots, a_m) \in D$ and $k = 1, \ldots, m$. A general procedura showing how to extend the function g to a function $\tilde{g} : \mathcal{D} \to \mathcal{F}_{LU}(\alpha_0, \ldots, \alpha_n)$, where $\mathcal{D} \subseteq \mathcal{F}_{LU}(\alpha_0, \ldots, \alpha_n)^m$ is a suitable domain, can be formulated within the two following steps:[2]

[2]Let us stress that a careful choice of the domain $\mathcal{D}$ has to be given to unsure the correctness of the extended mapping $\tilde{g}$.

1. Put $\mathbf{m} = \{1,\ldots,m\}$, consider $\pi : \mathbf{m} \to \{-,+\}$ and define

$$\mathbf{B}^{\pi(1),\ldots,\pi(m)} = \begin{pmatrix} f_{\alpha_0}^{\pi(1),\ldots,\pi(m)} & \cdots & f_{\alpha_n}^{\pi(1),\ldots,\pi(m)} \\ d_{\alpha_0}^{\pi(1),\ldots,\pi(m)} & \cdots & d_{\alpha_n}^{\pi(1),\ldots,\pi(m)} \end{pmatrix}, \tag{11}$$

where (for any $k = 0,\ldots,n$)

$$f_{\alpha_k}^{\pi(1),\ldots,\pi(m)} = g(f_{1\alpha_k}^{\pi(1)},\ldots, f_{m\alpha_k}^{\pi(m)}),$$
$$d_{\alpha_k}^{\pi(1),\ldots,\pi(n)} = g'_{x_1}(f_{1\alpha_k}^{\pi(1)},\ldots, f_{n\alpha_k}^{\pi(n)})d_{1\alpha_k}^{\pi(1)} + \cdots + g'_{x_n}(f_{1\alpha_k}^{\pi(1)},\ldots, f_{n\alpha_k}^{\pi(n)})d_{n\alpha_k}^{\pi(n)}.$$

2. Denote $\mathbf{B}_{\alpha_k} = (\mathbf{B}_{\alpha_k}^-, \mathbf{B}_{\alpha_k}^+)$ the pair of k-th columns of $\mathbf{B}^-$ and $\mathbf{B}^+$ and define

$$\tilde{g}(\mathbf{A}_1,\ldots,\mathbf{A}_m) = \mathbf{B} = (\mathbf{B}^-, \mathbf{B}^+) \tag{12}$$

such that, for any $k = 0,\ldots,n$, we have

$$(\mathbf{B}_{\alpha_k}^-, \mathbf{B}_{\alpha_k}^+) = (\min_{\pi:\mathbf{m}\to\{-,+\}} \mathbf{B}_k^{\pi(1),\ldots,\pi(m)}, \max_{\pi:\mathbf{m}\to\{-,+\}} \mathbf{B}_k^{\pi(1),\ldots,\pi(m)}), \tag{13}$$

where $\min$ (and analogously $\max$) is defined by

$$\min\left\{\begin{pmatrix} a \\ b\end{pmatrix}, \begin{pmatrix} c \\ d\end{pmatrix}\right\} = \begin{pmatrix} a \\ b\end{pmatrix} \textit{ if and only if } a \le c \textit{ or } (a = c \textit{ and } b \le d).$$

One can simply check that

1. $(\mathbf{A}^-, \mathbf{A}^+) + (\mathbf{B}^-, \mathbf{B}^+) = (\mathbf{A}^- + \mathbf{B}^-, \mathbf{A}^+ + \mathbf{B}^+)$,
2. $k(\mathbf{A}^-, \mathbf{A}^+) = (k\mathbf{A}^-, k\mathbf{A}^+)$ and $-k(\mathbf{A}^-, \mathbf{A}^+) = (-k\mathbf{A}^+, -k\mathbf{A}^-)$ for $k \ge 0$,
3. $\exp[(\mathbf{A}^-, \mathbf{A}^+)] = (\exp[\mathbf{A}^-], \exp[\mathbf{A}^+])$ with

$$\exp[\mathbf{A}^-]_{\alpha_k} = \begin{pmatrix} \exp[f_{\alpha_k}^-] \\ \exp[f_{\alpha_k}^-]d_{\alpha_k}^- \end{pmatrix} \textit{ and } \exp[\mathbf{A}^+]_{\alpha_k} = \begin{pmatrix} \exp[f_{\alpha_k}^+] \\ \exp[f_{\alpha_k}^+]d_{\alpha_k}^+ \end{pmatrix},$$

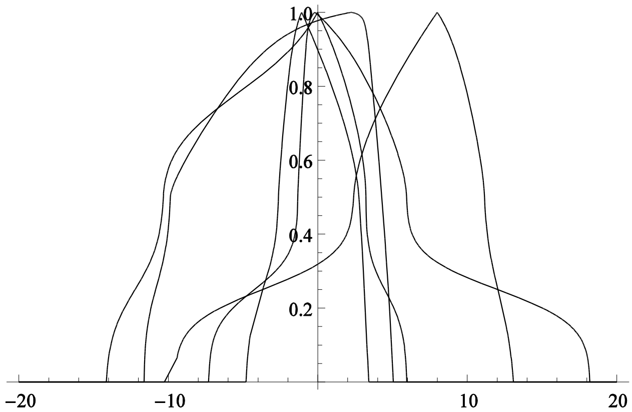
where the usual addition of matrices and the usual scalar multiplication are applied.

In order to define fuzzy random variables over LU-fuzzy numbers we follow the approach proposed by Kwakernaak (1978, 1979) and later formalized in a clear way by Kruse and Meyer (1987). Since each LU-fuzzy number is uniquely determined by a pair of matrices $(\mathbf{A}^-, \mathbf{A}^+)$, we will define fuzzy random variable using random matrices as follows. Let $(\Omega, \mathcal{A}, P)$ be a probability spaces. A mapping $X : \Omega \to \mathcal{F}_{LU}(\alpha_0,\ldots,\alpha_n)$ is said to be a *LU-fuzzy random variable* (or FRV for short), if there exist mappings $\mathbf{F}^-, \mathbf{F}^+, \mathbf{D}^-, \mathbf{D}^+ : \Omega \to \mathbb{R}^{n+1}$ such that $p_i \circ \mathbf{F}^-$, $p_i \circ \mathbf{F}^+$ and $p_i \circ \mathbf{D}^-$, $p_i \circ \mathbf{D}^+$, where p_i denotes i-th projection, are real-valued random variables for any $i = 1,\ldots,n+1$ and

$$X(\omega) = \left(\begin{pmatrix} \mathbf{F}^-(\omega) \\ \mathbf{D}^-(\omega)\end{pmatrix}, \begin{pmatrix} \mathbf{F}^+(\omega) \\ \mathbf{D}^+(\omega)\end{pmatrix}\right). \tag{14}$$

We say that two FRVs X and Y are *independent (identically distributed)*, if $p_i \circ \mathbf{F}_X^-$, $p_i \circ \mathbf{F}_X^+$, $p_i \circ \mathbf{D}_X^-$, $p_i \circ \mathbf{D}_X^+$ and $p_i \circ \mathbf{F}_Y^-$, $p_i \circ \mathbf{F}_Y^+$, $p_i \circ \mathbf{D}_Y^-$, $p_i \circ \mathbf{D}_Y^+$ are independent (identically distributed), respectively, for any $i = 1,\ldots,n+1$. On Fig. 1, we can see five pseudo-randomly generated LU-fuzzy numbers defined under the normal distribution

Figure 1: Pseudo-random LU-fuzzy numbers



Source: Authors' calculation in *Mathematica 8.0*; the kernels, ie. the points with the membership degree equal to 1, are determined from $N(0,4)$, further values are determined in such way that the difference between $f^{\mp}_{\alpha_i}$ and $f^{\mp}_{\alpha_{i+1}}$ are random values from $N(0,2)$, $d^{-}_{\alpha_i}$ and $-d^{+}_{\alpha_i}$ are determined from $U(0.01,2)$ for $\alpha_i \neq 1$ and $d^{-}_{1} = -d^{+}_{1} = 10000/x$, where x is a random value from $U(1,1000)$).

5. Comparative example

Let us assume a call and put option on German stock market index (DAX) on some day in February, 2006. For example, as Benko et al. (2007) documented for a two-weeks options, when fitting the BS model to market prices (in February 2, 2006) we can sometimes get different implied volatilities for the put and call options, especially if the moneyness is far from 1, ie. the option is either deep ITM or OTM. Here, we will calculate the fuzzy-value of the option on the basis of implied volatility obtained on a given day/month.

5.1 Option pricing model

Let σ_{LU} be an LU-fuzzy-random number defined on the basis of the crisp estimation of σ. Then we can model price returns by the fuzzy-stochastic model (8) above. In order to evaluate the risk-neutral expectation (3) via Monte Carlo simulation, ie. we put (8) into the exponential. Moreover, it is important to choose a proper mean-correcting parameter ω_{LU} such that the complex process (ie. including the riskless rate) will be martingale, when discounted by the riskless rate. Obviously, since (8) is an LU-fuzzy random process, parameter ω_{LU} has to be defined as LU-fuzzy number, too. Thus, we obtain:

$$S_T = S_t \exp[(r - \omega_{LU})\tau + \sigma_{LU}\sqrt{\tau}\varepsilon]. \tag{16}$$

Recall that ω_{LU} denotes a mean correcting parameter that compensates $\sigma_{LU}\sqrt{\tau}$. Therefore, in order to evaluate the model, we have to apply several operations with fuzzy numbers to obtain S_T. Finally, assuming a call option the exercise prices is deduced from S_T and after that the positive part of the matrix is returned as an option payoff for a given scenario. Obviously, in line with (3) we have to evaluate a huge number of such scenarios to obtain a reliable estimate of option price as a mean of matrices – we recall that the payoff for each particular scenario must be a positive LU-fuzzy number or zero matrix.

5.2 Results

Comparative results for call and put options are provided in Table 1. For simplicity and in order to increase the information value, we normalize the spot price to be 1, as well as the exercise price, ie. in the table we provide the ATM call and put option prices in percentage of the initial spot value. We keep the data similar to Benko et al. (2007) – riskless rate is 2.5% p.a., time to maturity is 10 business days, and implied volatility is 15.5% for both, the call and put (all these parameters are crisp numbers). The BS price is therefore 1.25 for the call and 1.10 for the put.

In order to get the option price, we have to basic choices, how to define LU-fuzzy random variable: (i) uniform distribution, (ii) normal distribution. In the first case, we spread the implied volatility by 50%, in the latter we set its 'variance' as 0.1. These parameters are estimated on the basis of long-run evolution of implied volatilities, ie. to cover a majority of values observed in the past.

Table 1: Call and put option 'mean' LU-fuzzy value

	Uniform distribution	Normal distribution
Call	$\left(\begin{pmatrix} 0.46 & 0.81 & 1.25 \\ 21.8 & 20.0 & 1134 \end{pmatrix}, \begin{pmatrix} 4.95 & 2.50 & 1.25 \\ -22.3 & -19.9 & -1134 \end{pmatrix}\right)$	$\left(\begin{pmatrix} 0.38 & 0.71 & 1.25 \\ 20.7 & 21.4 & 908 \end{pmatrix}, \begin{pmatrix} 5.97 & 3.16 & 1.25 \\ -20.7 & -20.9 & -908 \end{pmatrix}\right)$
Put	$\left(\begin{pmatrix} 0.04 & 0.35 & 1.10 \\ 22.3 & 20.0 & 1147 \end{pmatrix}, \begin{pmatrix} 2.29 & 1.66 & 1.10 \\ -21.8 & -20.2 & -1147 \end{pmatrix}\right)$	$\left(\begin{pmatrix} 0.06 & 0.35 & 1.10 \\ 20.7 & 20.7 & 909 \end{pmatrix}, \begin{pmatrix} 3.43 & 2.33 & 1.10 \\ -20.7 & -21.4 & -909 \end{pmatrix}\right)$

Source: author's calculations in *Mathematica 8.0*.

From the results depicted in Table 1, we can observe that both payoff functions lead to almost identical slopes of the resulting fuzzy-value and that the midpoints, ie. α-cut at the level 1, is very close to the BS price. Obviously, the midpoint is identical for both approaches, uniform and normal, though the slopes and left and right nodes can be slightly different.

Furthermore, we extend the analysis by considering one-year ATM call & put options. In this case, however, the market is not sufficiently liquid so that it can be difficult to detect the right interval. We therefore provide more extend analysis by considering several degrees of uncertainty – we assume only the case when volatility, the FRV, is based on normal distribution with standard deviation rising from 0.01 to 0.1 (Figure 2, first column); and FRV based on uniform distribution (Figure 1, second column).

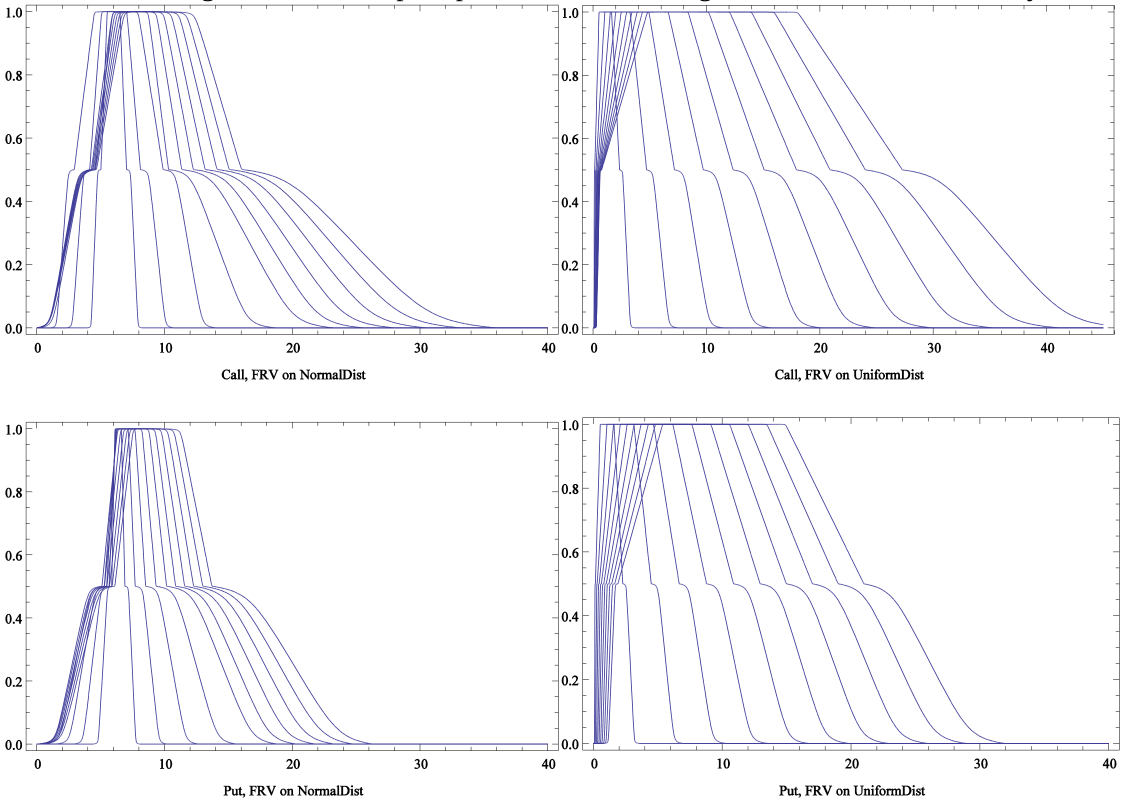
5. Conclusion

Volatility is one of the crucial parameters within option valuation problems. It is also the parameter, which is probably the most difficult to estimate. It is evident especially in the case of options with low liquidity or long maturity. In this paper, we have suggested a fuzzy random process to describe the volatility of log-returns underlying the option. Such approach allows us to keep more information about the possible option prices.

In particular, we have studied the case of call/put options on German DAX index, which allows us to utilize the implied volatility data. We have shown results that might be obtained when two different distributions are considered as a basis for calculation of the volatility intervals – the normal and uniform.

Within the further research on this topic we should concentrate in more details on the selection of various types of fuzzy-random numbers and their suitability for volatility modeling of particular option types.

Figure 1: Call and put options for various degree of fuzziness of volatility



Source: author's calculations in *Mathematica 8.0*.

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