MARKET RISK CAPITAL REQUIREMENT FOR INSURANCE COMPANIES BY COPULA APPROACH[1]

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Abstract

The Solvency of insurance companies means an ability to pay liabilities. From the point of view of supervisor bodies, the level of solvency of insurance companies is given by a Solvency II concept (compare to Basel II for banks). A novel approach of Solvency II is to take into account also market, credit, and operational risk, while two distinct requirements, the Minimum Capital Requirement (MCR) and the Solvency Capital Requirement (SCR) are distinguished. In this paper, we estimate both requirements for an internationally diversified equity portfolio of an insurance company. In order to model the portfolio evolution, multidimensional Lévy models coupled together by elliptical ordinary copula functions are assumed. We observe that elliptical (ie. symmetric) copula functions fail substantially in risk estimation over five day horizon. It results into 10% to 15% error in the estimation of the overall capital requirement.

Keywords: *Insurance company; multidimensional subordinated Lévy model; risk measuring; Solvency II*

JEL codes: **G21, G22,**

1. Introduction

For insurance industry, Solvency II presents the new rules of supervision that should be implemented in 2012. The existing legal regulation has some inadequacies – it is not as risk-sensitive as one would need and moreover, it does not involve the requirements for, say, market or operational risk. However, Solvency II is based on the risk based approach. That is, the insurance companies should take into account all the risks they can be exposed to and therefore, besides others, the entity has to take into account also the capital to cover market risk.

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A commonly adopted measure to express the market risk exposure is Value at Risk (VaR). Whereas this measure of risk is often criticised, see e.g. Artzner et al. (1999), many authors prefer to deal also with conditional Value at Risk. Among a large class of authors dealing either with portfolio risk estimation or Solvency II application or both, we can state the paper of Leibwein (2006), where calculation of capital requirement under the conditions of internal model in the context of Solvency II is described, or Rockafeller and Uryasev (2000), who considered the application of conditional Value at Risk for portfolio optimization. Moreover, Bec and Gollier (2009) recently started to research the term structure and cyclicity of Value at Risk and their consequences for the solvency capital requirement.

In this paper we aim at the estimation of risk of (international) equity portfolio held by an insurance company and express its impact on the overall capital it needs either in line with Solvency II approach or internal risk management purposes by means of VaR and cVaR.

We proceed as follows. In the following section, we introduce the concept of risk measuring in financial institutions and its relation to capital requirements. Next, in Section 3 the multidimensional subordinated Lévy model by terms of ordinary elliptical copula function is described. Finally, in Section 4 we calculate basic risk measures for international equity portfolio.

2. The new conception of risk management of insurance companies

Risk management is one of the most important activities of financial institutions. The role of risk management is to identify all risks a financial institution can be exposed to. Risk generally represents a measure of uncertainty. We can distinguish systematic risk and specific risk. The systematic risk, which can be also referred to as the market risk, is the risk that threats whole economy. It indicates that it is impossible to avoid it. On the contrary, the specific risk is a risk concerning only a particular company. Although it is usually more efficient to model just the systematic risk, any entity should take care of specific risk, too, especially if the portfolio is not full diversified.

There is a wide scale of financial institutions such as banks, investment funds, insurance companies and others. These all institutions follow a specific way of risk management. Generally, the risk management should improve a financial performance and allow a given entity to avoid unacceptable losses. The estimation of risk has to be regarded as a very challenging issue.

Any financial institution should manage risk to assure that it can be able to repay its liabilities. For banks, *capital adequacy* must be determined, while insurance companies have to be *solvent*. Some risk classes are typical for banks, others for insurance companies. Insurance companies have to deal with risks related with activities of insurance. That is, they have to face underwriting risk that is divided according to the activity character, i.e. life, non-life or health. By contrast, the operational and market risk are typical risk classes for both insurance and banking industry. However, these two classes of risk are the new elements of Solvency directive. Another important area in insurance risk management is asset/liability mismatch risk which is the risk resulted from time disharmony of assets and liabilities.

The credit risk, which is very significant for banks, is defined as a potential loss in consequence of counterparty default. Similarly, the operational risk is defined as a possibility of the loss that stems from failed internal processes, systems, people or external events. Finally, the market risk is the risk resulting from the fluctuation of market prices of assets. The exposure to market risk is determined by the impact of volatility of financial variables, such as equities, interest rates or FX rates.

In banking industry, a capital adequacy is determined according to Basel II, which is also called the New Accord. This document was signed in July 2004 and replaced Basel I. In Basel I accord the bank's capital adequacy was calculated with the help of Cooke Ratio. Following the new approach, the capital adequacy is determined by McDonough ration. The Basel II is based on three pillar approach. The first pillar contains the minimum capital requirements that take into account capital for credit risk, market risk and operational risk. The second pillar is focused on supervisory review and the last pillar includes market discipline to promote stability in the financial system.

In order to determine the capital charge for market risk either standard approach or internal model can be followed. Within the latter the risk is estimated on the basis of Value at Risk on the confidence level of 99% over ten days. Despite its drawbacks, Value at Risk is generally accepted measure of risk. It is defined as a maximum loss of portfolio for a given confidence level and time horizon. Value at Risk for probability level α is formally defined as

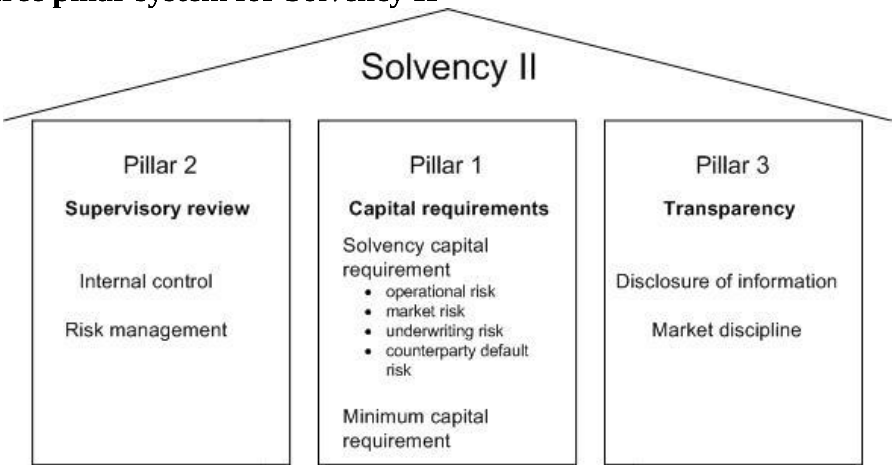
$$Pr\left(X \geq VaR_{\alpha,\Delta t}(X)\right) = \alpha \tag{1}$$

where X is a loss, $\alpha = 0.01$ and $\Delta t = 10$ days.

Recently, a directive similar to Basel II was adopted to measure the risk of insurance companies – we refer to it as the Solvency II. The Solvency II presents a new system of regulations for insurance companies within the European Union. It should replace present legal form that have some disadvantages such as, for example, insufficient risk sensitivity – it does not pay any attention to market or operational risk. These risks are new elements of the this system of regulation.

The Solvency II is based on three pillar system similarly as Basel II, see Figure 1. The first pillar contains requirements for technical provision and two levels of requirements, i.e. the solvency capital requirement and minimum capital requirement. The qualitative requirements and rules of supervision are included into the second pillar. Supervision authority obtains quality tools for assessing risk. The last pillar concerns publishing of substantial information to a supervision authority and to the public.

Figure 1 Three pillar system for Solvency II



The solvency capital requirement (SCR) reflects the capital that insurance company must hold to limit probability of default to 0.5 %. That is, Value at Risk measure with the confidence level of 99.5 % over one year is followed, and thus in (1):

$$Pr\left(X \geq VaR_{\alpha,\Delta t}(X)\right) = \alpha,$$

we assume $\alpha = 0.005$ and $\Delta t = 1$ year. By contrast, the minimum capital requirement (MCR) presents the level of capital under that policyholders' interests are threatened, so that in (1):

$$Pr\left(X \geq VaR_{\alpha,\Delta t}(X)\right) = \alpha,$$

we deal with $\alpha = 0.1$ and $\Delta t = 1$ year.

The Value at Risk is often criticized, because it does not fulfill the conditions of coherent measure of risk, see Artzner et al. (1999), particularly the subaditivity property.

Therefore, it is recommended to use other measures of risk, such as conditional Value at Risk sometimes called tail Value at Risk or Expected Shortfall. Value at Risk determines a probability with that a given loss can occur, but conditional Value at Risk determines the value of the average loss and is considered as a coherent measure of risk. Conditional Value at Risk is defined as expected loss exceeding Value at risk at the confidence level α and it is given

$$cVaR_{\alpha,\Delta t}(X) = E\left[X | X > VaR_{\alpha,\Delta t}(X)\right]. \tag{2}$$

3. Multidimensional Subordinated Lévy Models

The first focus at Lévy models with jumps goes back to 1930's. The most recent and complete monographs on the theory behind and/or application of Lévy models are Kyprianou et al. (2005), Applebaum (2004), Cont and Tankov (2004), Barndorff-Nielsen et al. (2001) and Bertoin (1998). However, a subordinated Lévy model, a rather non-standard definition of Lévy models as time changed Brownian motions, goes back to Clark (1973) or even Bochner (1949).

The motivation for the usage of subordinators and, simultaneously, the economic interpretation for estimated parameters for such a process is, that the information arrive to the market at random rate. Sometimes, there is a lot of important news within short time interval, sometimes the market is "sleeping". The variance of the subordinator should allow us to distinguish between markets with a relatively constant rate and very variable markets.

In this section, we first describe the marginal distributions of subordinated Lévy models. Then, we will show, how to obtain multidimensional distribution from marginal distributions by means of copula functions.[2]

3.1 Marginal Distribution

Generally, a Lévy process is a stochastic process, which is zero at origin, its path in time is right-continuous with left limits and its main property is that it is of independent and stationary increments. Another common feature is a so called stochastic continuity. Moreover, the related probability distribution must be infinitely divisible. The crucial theorem is the Lévy-Khintchine formula:

[2] For an alternative approach to building up multidimensional Lévy models, see e.g. Tichý (2008b).

$$\Phi(u) = \mathrm{i}\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} \left(\exp(\mathrm{i}ux) - 1 - \mathrm{i}ux\mathbb{I}_{|x|<1}\right)\nu(\mathrm{d}x). \tag{3}$$

For a given infinitely divisible distribution, we can define the triplet of Lévy characteristics,

$$\{\gamma, \sigma^2, \nu(\mathrm{d}x)\}.$$

The former two define the drift of the process (deterministic part) and its diffusion. The latter is a Lévy measure. If it can be formulated as $\nu(\mathrm{d}x) = u(x)\mathrm{d}x$, it is a Lévy density. It is similar to the probability density, with the exception that it need not be integrable and zero at origin.

Let X be a Brownian motion. If we replace standard time t in Brownian motion X,

$$X(t; \mu, \sigma) = \mu dt + \sigma Z(t), \tag{4}$$

by its suitable function $\ell(t)$ as follows:

$$X(\ell(t); \theta, \vartheta) = \theta\ell(t) + \vartheta Z\big(\ell(t)\big) = \theta\ell(t) + \vartheta\varepsilon\sqrt{\ell(t)}, \tag{5}$$

we get a subordinated Lévy model. Due to the simplicity (tempered stable subordinators with known density function in the closed form), the most suitable candidates for the function $\ell(t)$ seem to be either the variance gamma model – the overall process is driven by a gamma process from the gamma distribution with shape a and scale b depending solely on variance κ, $G[a, b]$, or normal inverse Gaussian model – the subordinator is given by an inverse Gaussian process based on the inverse Gaussian distribution, $IG[a, b]$.

The final step to get a model for a marginal distribution depends on the issue we are going to solve. For example, if the task is to model the prices of a financial asset, ie. strictly positive value, we should evaluate the Lévy model (5) in the exponential:

$$S(t) = S_0 e^{\mu t + X(\ell(t)) + \omega t}, \tag{6}$$

where μ states a long-term drift of the price (average return) and ω is the mean correcting parameter. By contrast, if we model a variable, which can be both positive and negative (eg. price returns), we can proceed as follows:

$$x(t) = \mu t - X\big(\ell(t)\big) - \theta t, \tag{7}$$

so that the long-term drift is fit again.

3.2 Copula functions

A useful tool of dependency modeling are the copula functions,3 i.e. the projection of the dependency among particular distribution functions into [0,1],

[3] In this paper, we restricted ourselves to ordinary copula functions. Basic reference for the theory of copula functions is Nelsen (2006), while Rank (2007) and Cherubini et al. (2004) target mainly on the application issues

$$\mathcal{C}: [0,1]^n \rightarrow [0,1] \text{ on } \mathbb{R}^n,\ n \in \{2,3,\dots\}. \tag{8}$$

Actually, any copula function can be regarded as a multidimensional distribution function with marginals in the form of standardized uniform distribution.

For simplicity, assume two potentially dependent random variables with marginal distribution functions F_X, F_Y and joint distribution function $F_{X,Y}$. Then, following the Sklar's theorem:

$$F_{X,Y}(x,y) = \mathcal{C}(F_X(x), F_Y(y)). \tag{9}$$

If both F_X, F_Y are continuous, a copula function $\mathcal{C}$ is unique. Sklar's theorem implies also an inverse relation,

$$\mathcal{C}(u,v) = \mathrm{F}_{X,Y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right). \tag{10}$$

Formulation (9) above should be understood such that the joint distribution function gives us two distinct information: (i) marginal distribution of random variables, (ii) dependency function of distributions. Hence, while the former is given by $F_X(x)$ and $F_Y(y)$, a copula function specifies the dependency, nothing less, nothing more. That is, only when we put both information together, we have sufficient knowledge about the pair of random variables X, Y.

Assuming that the marginal distribution functions of random variables are already known, the only further think we need to know to model the overall evolution is an appropriate copula function. With some simplification, we can distinguish copulas in the form of elliptical distributions and copulas from the Archimedean family. The main difference between these two forms is given by the ways of construction and estimation. While for the latter the primary assumption is to define the generator function, for the former the knowledge of related joint distribution function (e.g. Gaussian, Student-t, etc.) is sufficient.

It is therefore obvious that the *n*-dimensional subordinated Lévy model can be defined by terms of ordinary elliptical copula functions as follows:

$$F_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) = \mathcal{C}\left(F_1(x_1), F_2(x_2), \dots F_n(x_n)\right), \tag{11}$$

where $F_i(x_i)$ states for marginal distribution, ie. a suitable subordinated Lévy model, which can be different for particular i, and $\mathcal{C}$ is ordinary elliptical copula function.

3.3 Parameter Estimation of Multidimensional Subordinated Lévy Models

There exist three main approaches to parameter estimation for copula function based dependency modeling: exact maximum likelihood method (EMLM), inference for margins

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in finance. Alternatively, Lévy processes can be coupled on the basis of Lévy measures by Lévy copula functions. However, this approach is not necessary in our case.

(IFM), and canonical maximum likelihood (CML). While for the former all parameters are estimated within one step, which might be very time consuming, mainly for high dimensional problems or complicated marginal distributions, the latter two methods are based on estimating the parameters for the marginal distribution and parameters for the copula function separately. While assuming IFM, marginal distributions are estimated in the first step and the copula function in the second one, for CML instead of parametric margins empirical distributions are used. On more details see any of the empirically oriented literature such as Cherubini et al. (2004). In this paper we will assume IFM/CML approach.

Thus, we first estimate parameters of marginal distributions and then complete the dependency structure by the correlation matrix (Gaussian copula) and correlation matrix and degrees of freedom df (Student copula).

Since VG model can be treated as a Brownian motion (normal distribution) within a random (gamma) time, its density function can be formulated as follows, ie. Gaussian density evaluated in gamma time:

$$f_{VG}(x, g(t;\kappa);\theta,\vartheta) = \int_0^\infty \frac{1}{\vartheta\sqrt{2\pi g}} \exp\left(-\frac{(x-\theta g)^2}{2\vartheta^2 g}\right) \frac{g^{\frac{1}{\kappa}-1}\exp\left(-\frac{g}{\kappa}\right)}{\kappa^{\frac{1}{\kappa}}\Gamma\left(\frac{1}{\kappa}\right)} \mathrm{d}g. \tag{12}$$

In order to estimate the parameters by terms of MLM approach, (17) should be reformulated into, assuming discrete time step Δt:

$$f_{(X_{\Delta t})}(x) = \frac{2e^{\frac{\theta(x-\bar{\mu})}{\vartheta^2}}}{\vartheta\sqrt{2\pi}\kappa^{\frac{\Delta t}{\kappa}}\Gamma(1/\kappa)} \left(\frac{|x-\bar{\mu}|}{\sqrt{\frac{2\vartheta^2}{\kappa}+\theta^2}}\right)^{\Delta t/\kappa-1/2} K_{\Delta t/\kappa-1/2}\left(\frac{|x-\bar{\mu}|\sqrt{\frac{2\vartheta^2}{\kappa}+\theta^2}}{\vartheta^2}\right), \tag{13}$$

where $K_\eta(.)$ is a modified Bessel function of the third kind with index η:

$$K_\eta(x) = \frac{1}{2}\int_0^\infty y^{\eta-1} \exp\left\{-\frac{x}{2}(y^{-1}+y)\right\} \mathrm{d}y. \tag{14}$$

Moreover, setting $\bar{\mu} = \mu - \omega$ will allow us to estimate the real drift μ and the mean correcting parameter ω within one step.

4. Market risk estimation

Consider an insurance company that invested some fraction of available funds into foreign equity indices. The task is to estimate the risk of the portfolio over next 5 business days, including FX rate risk, by means of VaR and CVaR measures for $\alpha = \{0.1, 0.05, 0.01, 0.005, 0.001, 0.0003\}$. With respect to Solvency II, the most important significance is $\alpha = 0.005$ (SCR) and $\alpha = 0.1$ (MCR). Next, from the point of view of internal risk management, $\alpha = 0.0003$ might be also important – it can be regarded as an implied probability of default for, say, AA rating companies. Finally, $\alpha = 0.01$ relates to

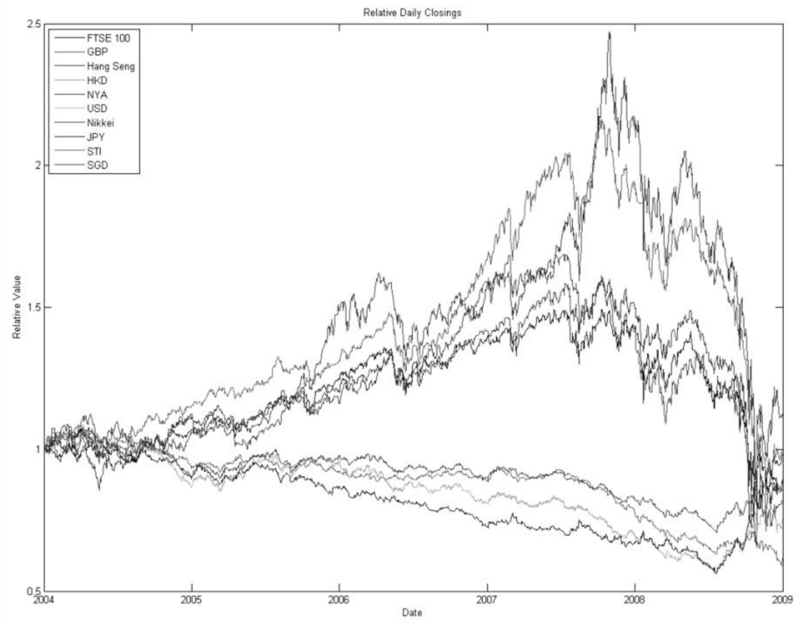
Basel II, ie. it is obligatory for banks and security firms, while higher values of α should help us with risk decay identification.

Besides VaR and cVaR, we will also report standard statistic of portfolio distribution, ie. maximal loss and gain, mean and median, standard deviation, and skewness and kurtosis. The distribution of portfolio returns will be estimated on the basis of n = 200 000 independent trials for five subsequent days of plain Monte Carlo simulation – plain Monte Carlo simulation can be regarded as a very effective approach when subordinated Lévy models are considered (see e.g. Tichý, 2008b).

4.1 Data

The data set consists of daily closing prices (taken over 2004–2008, ie. 1262 market quotes) of five equity indices all over the world, and thus denominated in five distinct currencies, FTSE 100 (London, GBP), Hang Seng (Hong Kong, HKD), NYA (New York, USD), Nikkei (Tokyo, JPI), and STI (Singapur, SGD). For simplicity, we normalized all data so that at the beginning (time zero, ie. the first quoted price) all indices start at the level of one, the same for FX rates. Thus, each foreign currency equals to one Czech koruna (CZK) at the beginning. The evolution of indices and FX rates in time is depicted in Figure 2.

Figure 2 Daily closing prices of indices and FX rates



Source: Author's calculation

From Figure 2 a relatively high correlation of particular indices and FX rates is evident. We can observe a strong price increase of all indices up to late 2008, with some drops in mid 2006. Next, there is a huge fall of all indices in 2008. Hence, as a result of the crisis, the closing prices in the end of the five years period are very close to the starting value. Concerning the evolution of FX rates, significant appreciation of CZK is apparent up to mid 2008. Then, with the change of risk perception by international investors, sharp depreciation arrives. These sudden and unanticipated changes of the market commonly cause the failure of Gaussian assumption – that is, high deviations from normal distribution, ie. excess kurtosis and nonzero skewness should be present. In appendix we present daily relative closings in CZK. It is apparent that the depreciation of CZK during 2008 did not compensate the fall of equity indices and the terminal value of the initial investment therefore tends to be lower.

Basic descriptive statistics (min and max return, mean and median, standard deviation, kurtosis and skewness) for particular indices and FX rates are depicted in Table 1. The first important conclusion is that daily returns are not significantly different to zero, as the median is. Next, standard deviations of equity indices are approximately twice as much as the deviations for FX rates. Also the kurtosis, ie. the probability of extreme returns, is generally higher for equity indices than for FX rates (except Japan). However, we cannot make any such statement for observed skewness.

Table 1 Basic descriptive statistics of log-returns for indices and FX rates

	FTSE 100	GBP	Hang Seng	HKD	NYA	USD	Nikkei	JPY	STI	SGD
Min	–0.093	–0.049	–0.136	–0.060	–0.102	–0.057	–0.121	–0.102	–0.092	–0.048
Max	0.096	0.040	0.142	0.048	0.115	0.043	0.137	0.055	0.075	0.036
Mean	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Median	0.000	0.000	0.000	0.000	0.000	0.000	0.000	–0.001	0.000	0.000
Standart deviation	0.013	0.006	0.017	0.008	0.014	0.007	0.016	0.009	0.013	0.006
Kurtosis	16.731	12.406	15.682	9.463	19.299	8.996	17.250	22.985	11.597	9.268
Skewness	0.035	–0.047	0.121	0.175	–0.400	0.207	–0.205	–0.549	–0.653	0.183

Source: Author's calculation

Let us assume the composition of the portfolio of insurance company as in Table 2.[4] Thus, the insurance company invests one third of available funds to US stocks (and USD) and only a small fraction to Hong Kong. From the other point of view, approximately one half is placed to the Anglo-Saxon markets (UK and US) and one half to Eastern Asia (Japan, Hong Kong, and Singapore).

Table 2 Portfolio composition

FTSE 100	16.12%
Hang Seng	5.00%
NYA	33.43%
Nikkei	26.18%
STI	19.27%

[4] Note, that the portfolio composition is equivalent to the efficient portfolio with minimal risk.

4.2 Marginal Distribution

We first focus on marginal distribution of particular risk drivers, ie. we will estimate the parameters of VG models assuming that there is no dependency. On the basis of (18) we apply MLM approach. The results for θ (the parameter to fit mainly skewness), κ (the parameter to fit mainly kurtosis, ie. random time arrival rate), ϑ (the parameter to fit mainly variance), and $\bar{\mu}$ (the drift including mean correction) are depicted in Table 3.

Table 3 Estimated parameters for marginal distributions

	θ	κ	ϑ	$\bar{\mu}$
FTSE 100	0.0000	4.8056	–0.0130	0.0000
GBP	–0.0002	0.7992	0.0056	–0.0002
Hang Seng	0.0002	4.4353	–0.0170	–0.0001
HSD	0.0003	1.6415	0.0084	0.0000
NYA	–0.0003	5.6673	–0.0140	0.0003
USD	0.0009	0.7587	0.0072	–0.0011
Nikkei	–0.0002	4.9777	–0.0160	0.0001
JPY	–0.0003	6.9244	–0.0090	0.0002
STI	–0.0008	1.3337	0.0121	0.0008
SGD	0.0007	0.6377	0.0058	–0.0008

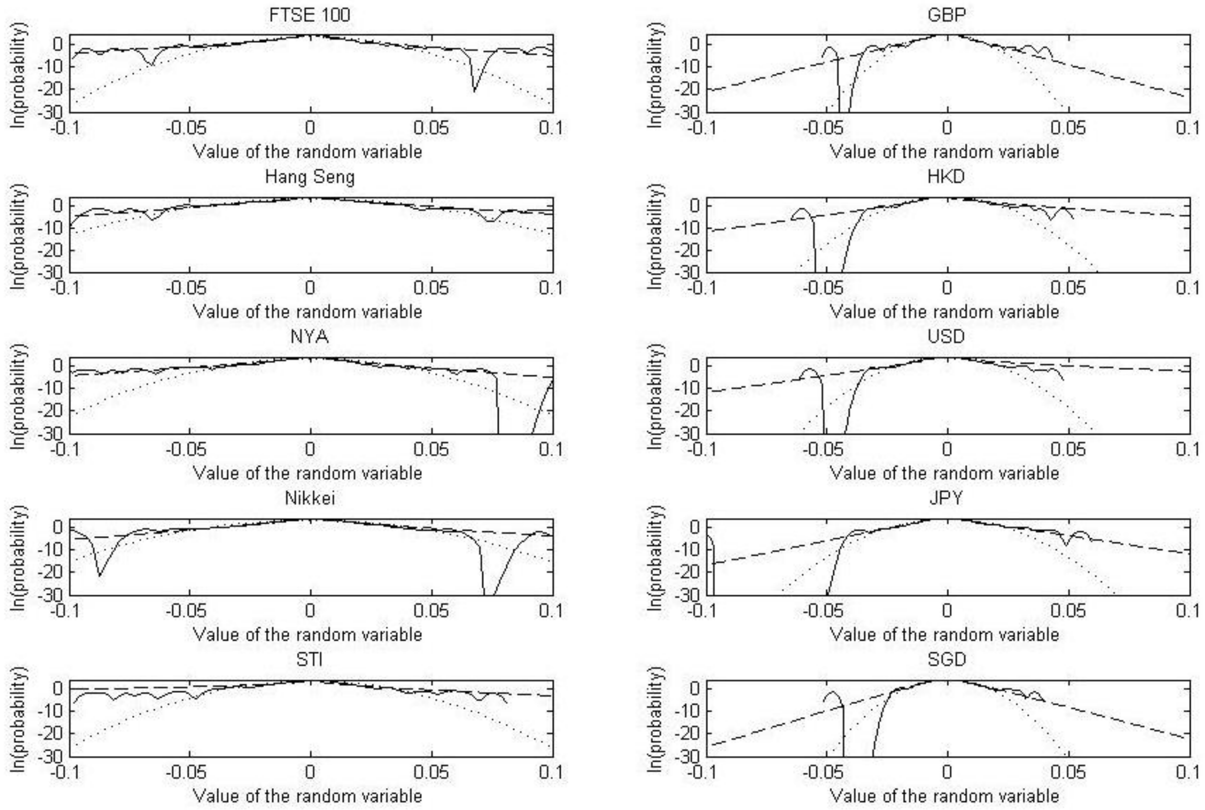
Source: Author's calculation

We can observe really various rates of random time arrival. We can neither say that e.g. equities require higher variance of the subordinator than FX rates, since there are two of both groups (STI and FTSE 100 for equities, USD and HSD for FX rates). Another point to note is the sign of θ. Generally, it is assumed that returns with negative skewness should require θ with a minus sign and vice versa – this rule of thumb is confirmed for all data.

VG model is an example of subordinated Lévy models, ie. it is driven by a subordinator, which should allow us to fit also higher moments of the distribution, skewness and kurtosis. Since we are interested in estimation of risk for left tails, we will examine the level of fitting by kernel function applied to log-densities (Figure 3). Within each chart, kernel function of empirical distribution is compared to the estimated VG distribution and normal distribution (Gaussian).

It is apparent that the decay of normal distribution is overall wrong. The probability of extremal returns decreases more than it should. It is stressed even further for FX rates since the estimation of their kurtosis is not very different to the ones of equities, but the variance is significantly smaller. Of course, the empirical kernel function is not very smooth due to the lack of extremal events, but it generally oscillates around the line given by the VG model.

Figure 3 Log-densities of marginal distributions



Source: Author's calculation

4.3 Linear Dependency – Gaussian Copula Approach

When Gaussian copula function is applied, we have to assume symmetric linear dependency among particular risk drivers, ie. no stressed dependency in tails. The only input factor is estimated correlation matrix, Table 4.

Table 4 Correlation matrix estimated for Gaussian copula

	FTSE 100	GBP	Hang Seng	HKD	NYA	USD	Nikkei	JPY	STI	SGD
FTSE 100	1.000	–0.039	0.354	–0.043	0.544	–0.051	0.350	–0.267	0.408	0.015
GBP	–0.039	1.000	0.004	0.574	0.003	0.579	0.029	0.395	0.010	0.618
Hang Seng	0.354	0.004	1.000	–0.184	0.208	–0.190	0.579	–0.289	0.700	–0.076
HSI	–0.043	0.574	–0.184	1.000	–0.102	0.974	–0.045	0.579	–0.132	0.871
NYA	0.544	0.003	0.208	–0.102	1.000	–0.099	0.168	–0.153	0.249	–0.061
USD	–0.051	0.579	–0.190	0.974	–0.099	1.000	–0.054	0.586	–0.132	0.893
Nikkei	0.350	0.029	0.579	–0.045	0.168	–0.054	1.000	–0.232	0.550	0.031
JPY	–0.267	0.395	–0.289	0.579	–0.153	0.586	–0.232	1.000	–0.297	0.623
STI	0.408	0.010	0.700	–0.132	0.249	–0.132	0.550	–0.297	1.000	–0.040
SGD	0.015	0.618	–0.076	0.871	–0.061	0.893	0.031	0.623	–0.040	1.000

Source: Author's calculation

Particular risk measures are depicted in Table 6 bellow: maximal loss and gain, mean median and standard deviation, skewness and kurtosis, VaR and cVaR for prespecified levels of confidence. All results are compared to the case, when no dependency among particular assets is assumed, and to the empirically obtained results.

Empirical results show us that maximal loss is higher than maximal gain. We can therefore assume that skewness should be negative and median should be higher than mean. However, symmetric Gaussian copula does not provide us a parameter to fit this asymmetry. Thus, the maximal loss is very close to the maximal gain, as well as the mean is close to the median, so that the skewness is almost zero. Another problem of Gaussian copula is that there is no way, how to fit the kurtosis. It results into high deficiency of particular risk measures of of VaR and cVaR, mainly those for higher degrees of sensitivity.

Next, we can examine the effect of dependency on the portfolio distribution. The dependency among particular returns formulated by the correlation matrix increases maximal loss and gain and variance, which is quite well matched by the Gaussian copula approach. No dependency in general also means no skewness and no excess kurtosis and lower levels of risky measures.

4.4 Stressed Tails of Risk Drivers' Distribution – Student Copula

By contrast to Gaussian copula function, the Student copula allows one to stress the dependency of extremal scenarios. That is, if things go very wrong (or good) within Gaussian copula approach, they go even further to the left (right) under Student copula approach. It is given by the fact that the dependency of extremal scenarios is higher as compared to normal ones. The estimated correlation matrix is depicted in Table 5, estimated degree of freedom is 4.28. If we compare particular correlations estimated for Student copula to the ones, we have already estimated for Gaussian copula, there are only a few, and rather insignificant differences.

Table 5 Correlation matrix estimated for Student copula, *df* = 4.28

	FTSE 100	GBP	Hang Seng	HKD	NYA	USD	Nikkei	JPY	STI	SGD
FTSE 100	1.000	–0.064	0.331	–0.015	0.565	–0.022	0.315	–0.206	0.396	0.018
GBP	–0.064	1.000	–0.018	0.570	–0.017	0.575	–0.007	0.438	–0.027	0.620
Hang Seng	0.331	–0.018	1.000	–0.156	0.182	–0.163	0.558	–0.221	0.669	–0.068
HSI	–0.015	0.570	–0.156	1.000	–0.082	0.974	–0.023	0.585	–0.115	0.884
NYA	0.565	–0.017	0.182	–0.082	1.000	–0.083	0.152	–0.130	0.244	–0.054
USD	–0.022	0.575	–0.163	0.974	–0.083	1.000	–0.029	0.586	–0.114	0.905
Nikkei	0.315	–0.007	0.558	–0.023	0.152	–0.029	1.000	–0.165	0.565	0.030
JPY	–0.206	0.438	–0.221	0.585	–0.130	0.586	–0.165	1.000	–0.243	0.656
STI	0.396	–0.027	0.669	–0.115	0.244	–0.114	0.565	–0.243	1.000	–0.039
SGD	0.018	0.620	–0.068	0.884	–0.054	0.905	0.030	0.656	–0.039	1.000

Source: Author's calculation

The risk measures are depicted once again in Table 6. Comparing the results among Gaussian copula and Student copula approaches, the stressed dependency in tails of the distribution implies higher kurtosis, (quite surprisingly) slightly higher variance – now, very close to the empirical one, higher maximal loss and gain and, as we could suppose, higher measures of risk, except VaR for α = 10%. Obviously, as we go further to the tail (it should be valid for both tails), the difference increases.

Table 6 Risk parameters for Gaussian and Student copula function

Measure		No dependency	Gaussian	Student	Empirical
max loss		–10.859%	–13.854%	–21.927%	–20.434%
max gain		11.607%	14.274%	21.239%	14.171%
Mean		–0.067%	–0.080%	–0.090%	–0.165%
Median		–0.065%	–0.076%	–0.084%	0.096%
standard deviation		1.791%	2.373%	2.437%	2.476%
Kurtosis		3.737	3.734	5.149	10.525
skewness		–0.005	–0.007	–0.033	–1.022
VaR	10%	2.289%	3.003%	2.960%	3.243%
	5%	3.004%	3.946%	4.007%	4.331%
	1%	4.473%	5.982%	6.466%	6.527%
	0.5%	5.092%	6.762%	7.582%	10.367%
	0.1%	6.417%	8.664%	10.392%	16.662%
	0.03%	7.481%	9.884%	12.800%	20.434%
cVaR	10%	3.265%	4.311%	4.492%	4.959%
	5%	3.919%	5.196%	5.562%	6.215%
	1%	5.335%	7.123%	8.166%	10.861%
	0.5%	5.923%	7.911%	9.368%	13.951%
	0.1%	7.286%	9.674%	12.296%	20.434%
	0.03%	8.292%	10.863%	14.402%	—

Source: Author's calculation

Student copula is still a symmetric one. Therefore, the maximal likelihood approach we have chosen to estimate the parameters of the model, namely degrees of freedom, tries to compensate errors in both tails. Since the empirical distribution of the portfolio is highly negatively skewed, the kurtosis of the estimated model is not very close to the one we obtained empirically. It also results into the fact that we are not able to fit VaR and cVaR measures for far left tails.

4.5 The Implication for Capital Requirements of Insurance Company

The financial crises and subsequent evolution at financial markets during the last year or two has persuaded insurance companies to significantly decrease the investments into equities. We will therefore assume the balance sheet of 2006, ie. the investments into equities are about 10% of total financial placement. As we stated in Section 2, the market risk capital requirements consists of interest rate, equity, property, spread, concentration, and currency risk. Since insurance companies usually hedge themselves against the evolution of foreign currencies and the counterparties for debt instruments are generally with very high creditworthiness, equity risk usually gives rise to approximately one third of the market risk capital requirement (including diversification among risk classes). Since we do not assume FX rate hedging of our portfolio, it is reasonable to assume that estimated VaR (cVaR) represent up to 50% of the market risk capital requirement. In average, we might assume that it will be about 20% of the overall capital requirement. It means that the error of 76%/50% (Table 6, assuming Gaussian/Student copula and empirical results for $\alpha = 0.5\%$) would result into 15%/10% error in the overall capital as based on both VaR and cVaR.

The largest Czech insurance companies can invest into equities even several hundreds of millions. Assume for simplicity that the portfolio is worth 500 mil. CZK. According to the results in Table 6 it gives us approximate SCR (over 5 days) at the level of 35 mil. CZK (for copula models), which can be compared to empirical results (51 mil. CZK). However, we can assume that the results would be more close for one year horizon (kurtosis of yearly data is not so high), ie. 270 mil. CZK for Student copula (ie. VaR/cVaR over 5 days times square root of time and correction factor) should be treated as a more reasonable result. The results we achieved for a given set of data therefore suggest us several interesting directions for further research, namely (mixed) Archimedean copula family application.

5. Conclusion

Coherent risk modeling and its subsequent management is a challenging task for risk units of each financial institution. In this paper, we have focused on its modeling from the point of view of an insurance company, in particular the market risk of equity investments that can in average comprises about 10 of overall financial placements.

In order to estimate the risk measures of VaR and cVaR for a set of significance levels, we have assumed that the financial returns follow multivariate subordinated Lévy model (VG process), ie. we coupled marginal distributions together either by Gaussian or Student copula functions, which are two examples of ordinary elliptical copula functions.

It was document that the Student copula approach provides us with much better result than standard Gaussian copula assumption. However, through a very high kurtosis and asymmetry in portfolio returns (ie. stressed left tails), it is still significantly far from empirically obtained results. There is therefore strong implication for optimal capital estimation – copula functions with asymmetry in tails should be examined.

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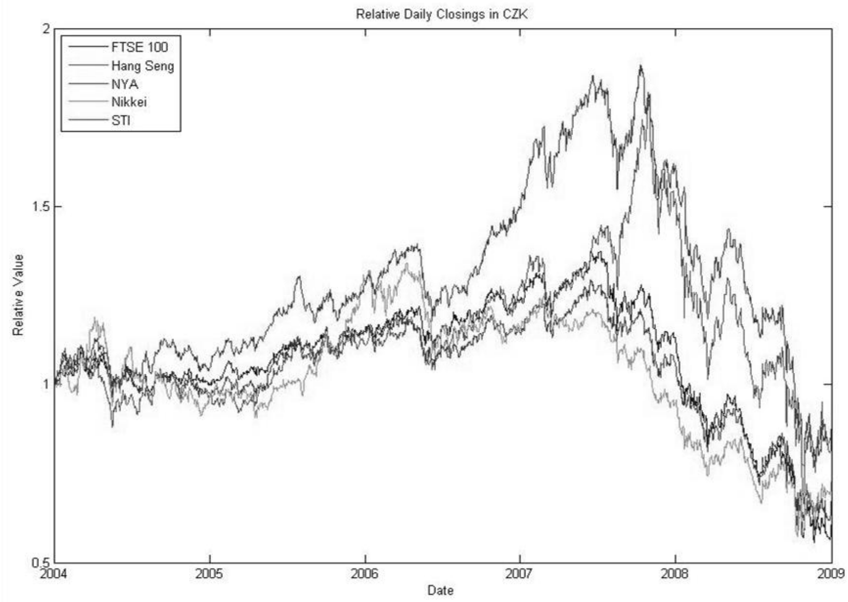
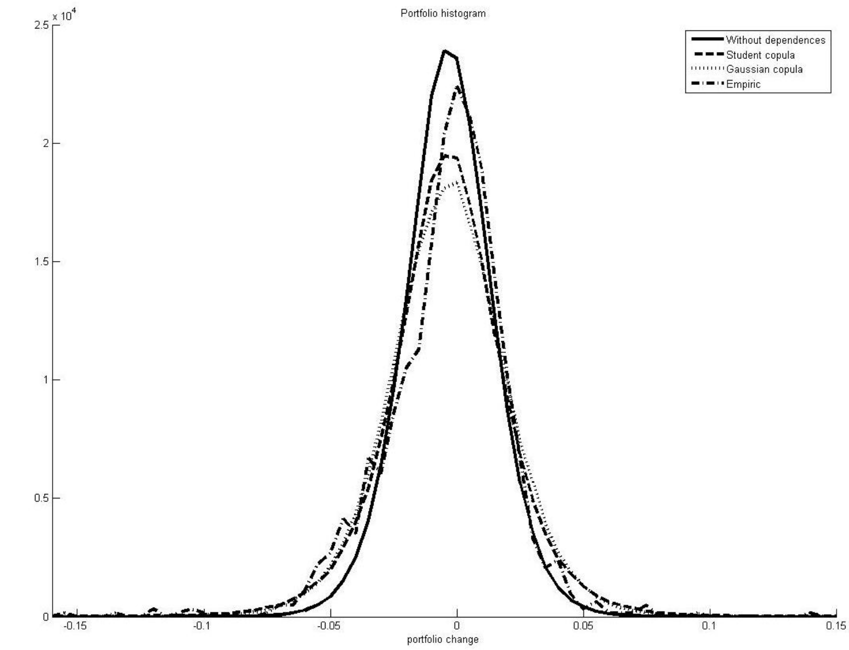
APPENDIX

Figure 3 Relative daily closing prices (in CZK)



Source: Author's calculation

Figure 4 Estimation of PDF for particular models of returns



Source: Author's calculation