Some Results on the Law of Large Numbers for Random LU-Fuzzy Numbers in the Context of Simulation of Financial Quantities

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Abstract
Financial problems can be analyzed by several ways. One of them is based on simulation of scenarios that might happen. In the standard case, we assume that a given financial quantity in question is a stochastic (or random) variable, ie. it follows some probability distribution and its possible states can get prescribed particular probabilities. The analysis is basically related to the Law of Large Numbers, which allows one to show that as the number of scenarios approaches to infinity, the modeled results should approaches to the reality. However, in financial modeling it can appear that the estimation of parameters of such models (e.g. volatility). One can recognize an unnatural simplification of parameters that can lead to a loss of important information hidden in data. If one wants to apply Monte Carlo simulation to analyze a financial problem with values expressed by imprecisely defined numbers, it is important to show that random variables with imprecisely defined numbers satisfy the strong law of large numbers, as well. Otherwise such approach would have no sense.
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1. Introduction

Solutions to most of the problems of financial modeling require that the law of large numbers holds. It concerns financial risk issues as well as portfolio selection problem. The law of large numbers indicates that as the number of samples used to make decision increases the mean of the sample approaches the mean of the whole population. One may also assume that if the mean approaches, it is likely that other properties of the sample will approach to those of the population as well.

Law of large numbers has already been quite well studied for the case of random variables in finance, ie. financial quantities following stochastic processes. However, in some cases available information does not provide us justification to use probability distribution and assume that a probability can be assigned to each possible state.

Instead, we might consider utilization of the fuzzy set theory, that is, possible outcomes are described in the form of a fuzzy number and its α-cuts. Obviously, such type of a number is just a part of potentially very complex system used to analyze a given financial modeling problem and various computational operations can be required. In order to simplify these operations, several types of approximations to fuzzy numbers have been suggested.

Among the simplest cases belong triangular and rectangular approximations, though the result can be quite far from Zadeh's extension principle. As an alternative, which is still computationally simple but simultaneously provides high degree of robustness, we can consider the approximation to LU fuzzy numbers by rational splines, which is based on advanced system of parameters.

Clearly, before using such type of approximation within financial modeling, we should examine if the necessary conditions, such as the law of large numbers, still holds. Thus, in this paper we provide some preliminary results on the validity of the strong law of large numbers to the approximation of LU fuzzy numbers by rational splines.

We proceed as follows. In the next section we show some consequences of the law of large numbers (LLN) in finance. Subsequently, basic conception of fuzzy set theory, including random LU-fuzzy numbers is provided. Next, previous results on LLN for fuzzy numbers are briefly reviewed and original theorem on LLN for parametrized approximation of random LU-fuzzy numbers is suggested. Finally, the theoretical result is confirmed by an illustrative example.

2. The consequence of law of large numbers in finance

Following the law of large numbers (LLN) one can estimate particular moments of the probability distribution using sufficiently large number of observations – realizations of random variable X. The justification lies in the fact that with increasing number of observations the sequence of random variables X_1, X_2,… converges to random variable X. If the random variable converges, it is natural that its moments must converge too, which in turn allows us to obtain particular parameters of probability distribution.

Formally, the strong law of large numbers can be written as follows:

$$P(X_n = X|n \to \infty) = 1 \tag{1}$$

and can be read as that the sequence of random variables X_n converges almost surely to random variable X as n approaches to infinity.

Alternatively, one can consider the weak law of large numbers, which is based on the difference:

$$P(X_n - X \geq \epsilon|n \to \infty, \epsilon > 0) = 0. \tag{2}$$

A natural consequence of LLN is Monte Carlo simulation, a powerful tool of financial modeling suitable for analyzing of many complex problems in economics and finance, such as pricing of options with complex payoff function (ie. exotic options) or modeling based on complex underlying processes. The principles of the methodology are based on the Law of large numbers – with increasing number of realizations of random variable the characteristics of the sample (ie. mean, standard deviation, etc.) will approach to the assumed values. The same is true for the approximated density function. Obviously, the realizations of the random variable can be obtained either on the basis of observations of real event or artificially, though, only the second option allows one to get really huge number of realizations and thus approach to the assumed values.

We can say that the first step of the methodology is to produce (generate) vector of random variates ε with a given dimension. Next, we evaluate particular function f(ε) that sufficiently describes (approximate) the functional relation between the randomness and the object of the study. Final step is the evaluation, which can be, for example, calculation of a given moment of the probability distribution, quantile estimation (useful for risk measure issue, such as Value at Risk), estimation of the probability, that particular event will happen, such as option exercising, and many others.

Law of large numbers lies, for example, also behind the portfolio theory – as we increase the number of risky assets with correlation $\rho < 1$ in the portfolio to infinity, the idiosyncratic risk vanishes. The reason is

The risk reduction is utilized also within credit risk management and in insurance industry.

3. Fuzzy numbers, theirs approximations and random LU-fuzzy variable

The existence of vague conditions in many (financial) decision-making problems led to a generalization of a classical *set theory*, where the membership of a particular element is a binary variable, ie. it either belongs to the set or not. By contrast, a *fuzzy set* theory provides much more freedom when the membership of a particular element is assigned. In particular, the convention is to use a membership degree from close interval [0,1] and obviously, zero membership corresponds to 0 within classical set theory (the item does not belong to the set for sure), while unit membership means the contrary.

Clearly, in practical applications we do not use sets in their broad definitions, but rather real numbers; similarly, a special case of a fuzzy set is called a *fuzzy number*. A fuzzy number is a mapping $A : R \to [0,1]$, where R denotes the set of all real numbers. It is a special case of a fuzzy set, which is convex, $A(\lambda x + (1-\lambda)y) \geq \min(A(x), A(y))$ for any $x, y \in R$, $\lambda \in [0, 1]$, normal, $A(x_0) = 1$ for a suitable $x_0 \in R$, upper semicontinuous and supp(A) is bounded, where supp(A) $= \mathrm{cl}\{x > R \mid A(x) > 0\}$ and cl is the closure operator.

A fuzzy number is commonly represented in terms of α-cuts. Since we have already stated that a fuzzy number is an upper semicontinuous real function, it is sufficient to replace each α-cut by its endpoints. In particular, we use u_α^- for the left endpoint, while u_α^+ states the right endpoint. It follows that a fuzzy number can be described by two functions u and v, $,v: [0,1] \to R$ $(u^-(\alpha)u_\alpha^-$ and $u^+(\alpha)u_\alpha^+)$, such that:

1. u^- is a bounded monotonic non-decreasing function which is left-continuous on (0;1] and the right-continuous for $\alpha = 0$,
2. u^+ is a bounded monotonic non-increasing function which is left-continuous on (0;1] and the right-continuous for $\alpha = 0$,
3. $u^-(\alpha) \leq u^+(\alpha)$ for any $\alpha \in [0,1]$.

Since evaluation of various arithmetic operations with fuzzy numbers can be complicated, one can use their parametric representation as proposed by Stefanini et al. (2006), instead.

In the past, the most popular models of fuzzy numbers were the triangular and trapezoidal models investigated by Dubois and Prade in (1978), which came from quite simple rules on basic calculation. However, a more advanced model of fuzzy numbers, which has been suggested relatively recently by Guerra and Stefanini in (2005), is based on the interpolation of particular knots using rational splines. Such approach generalizes the triangular fuzzy numbers and gives a broad variety of shapes enabling more precise representation of fuzzy real data keeping the requirements on the calculation procedures still simple.

Stefanini et al. (2006) also show that each LU-fuzzy number can be approximated with an arbitrary precision by a parametrized LU-fuzzy number and that the quality of approximation can be kept even after arithmetic operations.

If we partition the unit interval by $i = 0, 1, \ldots, m$, we can express the parametrized LU fuzzy number by the following matrix:

$$A = \begin{pmatrix} (u_0^-, \delta u_0^-) & \cdots & (u_m^-, \delta u_m^-) \\ \left(u_0^+, \delta u_0^+\right) & \cdots & \left(u_m^+, \delta u_m^+\right) \end{pmatrix}. \tag{3}$$

where u states particular data with natural relation

$$u_0^- \leq u_1^- \leq \cdots \leq u_m^- \leq u_m^+ \leq \cdots \leq u_1^+ \leq u_0^+, \tag{4}$$

and δu describes particular slopes (derivative at u), which means that δu_0^- is always nonnegative, while δu_0^+ nonpositive.

In Stefanini et al. (2006), the authors proposed to defined random prametrized LU-fuzzy numbers in such a way that both the data and slopes are random variables satisfying all conditions stated for the parametrized LU-fuzzy numbers (see (3)). Similarly, we can defined a random matrix (i.e., its values are random variables) which values express parametrized LU-fuzzy numbers.

Thus, having a probability space (Ω, A, P) and random variables X_i^ι, δX_i^ι, with $i = 0, 1, \ldots, m$ and $\iota \in \{-,+\}$, we can write parametrized random LU-fuzzy number as follows:

$$X = \begin{pmatrix} (X_0^-, \delta X_0^-) & \cdots & (X_m^-, \delta X_m^-) \\ \left(X_0^+, \delta X_0^+\right) & \cdots & \left(X_m^+, \delta X_m^+\right) \end{pmatrix}, \tag{5}$$

with conditions similar to (3):

$$X_0^-(\omega) \leq X_1^-(\omega) \leq \cdots \leq X_m^-(\omega) \leq X_m^+(\omega) \leq \cdots \leq X_1^+(\omega) \leq X_0^+(\omega) \tag{6}$$

and nonnegativity / nonpositivity of $\delta X_i^\iota(\omega)$ that must hold for any ω from the space.

An important concept of random variables is their independence and that they are identically distributed. Obviously, even in the case of (parametrized) random LU-fuzzy numbers we can define similar condition.

We say that two parametrized random LU-fuzzy numbers X and Y are *pciid random variables* if X and Y are *partially cut-wise independent and simultaneously partially cut-wise identically distributed random variables*, where X and Y are cut-wise independent random variables if both, the data and the slopes, ie. X_i^ι, Y_i^ι and δX_i^ι, δY_i^ι, are simultaneously independent random variables and similarly X and Y are cut-wise identically distributed random variables if both, the data and the slopes are simultaneously identically distributed random variables

4. Previous results on LLN

Since fuzzy sets, as well as their special cases an LU-fuzzy numbers can be regarded as generalization of real numbers, also parametrized approximations to random LU-fuzzy numbers are generalizations of random variables. It is therefore natural that various variants of the law of large numbers might have their counterparts in the theory of random fuzzy numbers.

The concept of fuzzy random variables was firstly introduced by Kwakernaak (1979), though Stein and Talati (1981) and Puri and Ralescu (1986) suggested alternative definitions.

The strong law of large numbers for fuzzy random variables due to Kwakernaak (1979) was proved by Kruse (1982), while Klement et al. (1986) provided some related limit theorems. By contrast, Miyakoshi and Shimbo (1984a,b) obtained a strong law of large numbers for independent fuzzy random variables and also generalized Birkhoff's ergodic theorem to fuzzy random variables. Furthermore, Jang and Kwon (1996) focused specifically on uniform strong law of large numbers in the presence of fuzziness. Besides that, Taylor (1997) and Joo (2004) focused on weak law of large numbers. Recently, opposed to original approach to LLN, which assumed usage of Zadeh's extension principle to calculation of the average, Terán (2013) used a general t-norm extension principle based on a continuous triangular norm.

5. Strong law of large numbers and LU fuzzy numbers

Before proceeding further, let us define the Hausdorff distance between two arbitrary LU-fuzzy numbers A and B, $A = \{[u_\alpha^-, u_\alpha^+]\}_{\alpha\in[0,1]}$ and $B = \{[v_\alpha^-, v_\alpha^+]\}_{\alpha\in[0,1]}$:

$$d_H(A,B) = \vee_{\alpha\in[0,1]} \max\left(\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right). \quad (7)$$

This measure allows us to evaluate the distance between two LU-fuzzy numbers; in other words, we compare them on the basis of their distance, which is approach similar to the distance of real numbers measured by absolute value.

Assuming parametrized random LU-fuzzy number we can formulate a theorem on the strong law of large numbers as follows, utilizing the fact that if each component of the random matrix has a finite expected value than a parametrized random LU-fuzzy number has a finite expected value.

Theorem 1. Let (Ω, A, P) be a probability space and $\{X_i\}$ be a sequence of cut-wise independent and identically distributed parametrized random LU-fuzzy numbers with the finite expected value $E[X_1]$. Then,

$$d_H\left(\frac{1}{n}\sum_{j=1}^{n} X_j, \mathrm{E}[X_1]\right) \to 0 \quad a.s. \quad (8)$$

The proof is based on the fact that replacing $\frac{1}{n}\sum_{j=1}^{n} X_j$ by A_n and $\mathrm{E}[X_1]$ by B the strong law of large numbers is first applied to the components of the random matrix X (ie. it is applied to random variables). After some operations applied to particular components and subsequent manipulations it can be easily shown that

$$d_H(A_n(\omega), B) = \bigvee_{\alpha \in [0,1]} \max\left(\left|A_{n\alpha}^{-}(\omega) - B_{\alpha}^{-}\right|, \left|A_{n\alpha}^{+}(\omega) - B_{\alpha}^{+}\right|\right) \leq \epsilon' \leq \epsilon \quad (9)$$

for any $n_{\epsilon'} > n$.

6. Illustrative example

Let first start with random variables and the implication of LLN. Following Tichý (2010), we can assume Monte Carlo simulation of random variables from normal distribution, $\varepsilon \in N[0,1]$, via *MersenneTwister* approach. Following Table 1 shows how particular descriptive statistics (mean, variance, skewness, kurtosis) approach to their assumed values (first row) with increasing number of scenarios (*N*).

Table 1: Title of the Table (Times New Roman, 11, centered)

N (Scenarios)	Mean (0)	Variance (0)	Skewness (0)	Kurtosis (3)	Time
10	−0,0593	0,7997	0,4586	2,8424	0,000
100	0,013	1,0484	0,2829	3,5538	0,000
1 000	0,0277	1,0777	0,0977	3,2436	0,000
10 000	−0,0030	1,0014	−0,0122	2,9165	0,015
100 000	−0,0008	1,0037	−0,0074	2,9793	0,032
1 000 000	−0,0014	1,0043	0,0056	2,9892	0,296
10 000 000	0,0004	0,9998	−0,0003	2,9989	2,434

Source: author's calculations, Tichý (2010)

Variables from normal distribution are a crucial part of most of the financial models. For example, we can state a Wiener process *Z*(*t*):

$$Z(t) = \sigma\sqrt{t}\varepsilon \quad (10)$$

where ε is a random variable from the normal distribution, *t* is the time passage and σ is the volatility.

In standard definition of (10) the volatility is supposed to be a constant. However, in some problems, such as pricing of options on illiquid underlying assets, it is not possible to get a reliable estimation of it. Therefore, in Holčapek and Tichý (2012) we have suggested to replace deterministic σ by LU-fuzzy number as follows:

$$X(t) = \sigma_{LU}\sqrt{t}\varepsilon. \quad (11)$$

In fact, in this way we have obtained random LU-fuzzy variable by multiplying LU-fuzzy number with random variable from normal distribution and furthermore, we have put it into the exponential form in order to model a stock price and subsequently price the option with riskless return *r*, time to maturity τ and mean correcting parameter in the form of expected LU-fuzzy number ω_{LU}:

$$S_T = S_t \exp\left[(r - \omega_{LU}) + \sigma_{LU}\sqrt{\tau}\varepsilon\right]. \quad (12)$$

Thus, let us assume and ATM option with time to maturity of one year, underlying asset price 100, and volatility specified as LU-fuzzy number around 0.15 with spread parameter 0.1. Expected LU-fuzzy payoff of the option is obtained with 1 000 000 simulations of ε is as follows:

$$\begin{pmatrix} (1.22,227)\ (3.10,209)\ \ (6.76,7023) \\ (22.56,-138)\ (14.66,-161)\ (6.76,-7023) \end{pmatrix},$$

and the central value can be regarded as identical with the value obtained due to Black and Scholes model (1973), ie. assuming (12) with crisp parameter of volatility. Thus, we empirically show that with a sufficiently large number of observations, we can approach to assumed mean even with a parametrized approximation of random LU-fuzzy number.

7. Conclusion

Solving of several financial problems requires combination of random variables and fuzzy numbers. The analysis of random variables implicates the law of large numbers, an inevitable condition of eg. Monte Carlo simulation. A similar rule is required in order to apply the simulation approach to random fuzzy numbers. In this paper, we have analyzed parametrized approximations to random LU-fuzzy numbers in more details and thus we confirmed the reliability of the estimates of random experiments both theoretically and empirically.

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